Journal of Engineering Physics and Thermophysics, Vol. 70, No. 5, 1997

APPLICATION OF THE METHOD OF INTEGRAL CROSS-SECTIONS FOR ESTIMATING THE EFFECTIVE ELASTIC PROPERTIES OF COMPOSITE MATERIALS

V. V. Novikov and O. B. Papkovskaya

UDC 518.12:536.19

We substantiate estimates of the upper and lower bounds of the effective elasticity modulus of piecewise homogeneous bodies. Using the method of integral cross-sections, we solve the elastic problem concerning the bending of a Kirchhoff inhomogeneous square plate and find the effective cylindrical stiffness of this inhomogeneous plate for various geometric parameters of inclusion, as well as determine the upper and lower bounds of the effective stiffness for the set of problems considered.

Introduction. Inhomogeneous materials such as composite, granular, and fibrous materials, various alloys, metal ceramic, etc., have found a wide application in modern technology. There are a number of methods that permit one to investigate the physical properties of these materials with a complex form of inclusion and take into account end effects and the geometry of the inhomogeneities. These are the methods of R-functions [1], averaging [2], finite elements [3] integral cross-sections [4], and others.

In the present work we solve elasticity problems by using the method of integral cross-sections and obtain estimates of the upper and lower bounds of the elastic properties of microinhomogeneous materials (MIM). Basic concepts and the essential idea of the method are discussed in more detail in [5], where a numerical technique for calculating temperature fields in piecewise homogeneous bodies was developed and implemented. The present work is a further development of the method of integral cross-sections; it suggests dependences and relations for the set of elasticity problems considered that can be used in engineering and other computations.

Method of Integral Cross-Sections. When determining the estimates of the upper and lower bounds of the elastic properties of microinhomogeneous materials by means of the method of integral cross-sections, use can be made of the inequalities obtained by Hill [6, 7]:

$$\sigma_{kl}^{(A)}\,\varepsilon_{kl}^{(A)} \le \sigma_{kl}^{(B)}\,\varepsilon_{kl}^{(B)} + 2\sigma_{kl}^{(A)}\,(\varepsilon_{kl}^{(A)} - \varepsilon_{kl}^{(B)})\,, \quad \sigma_{kl}^{(A)}\,\varepsilon_{kl}^{(A)} \le \sigma_{kl}^{(B)}\,\varepsilon_{kl}^{(B)} + 2\varepsilon_{kl}^{(A)}\,(\sigma_{kl}^{(A)} - \sigma_{kl}^{(B)})\,,$$

where $(\varepsilon_{kl}^{(A)},\ \sigma_{kl}^{(A)})$, $(\varepsilon_{kl}^{(B)},\ \sigma_{kl}^{(B)})$ are two arbitrary pairs of functions that characterize two states of an elastic body. For convenience in further formulations we shall introduce the operators of averaging over the coordinates:

$$\{f(\mathrm{r})\}_L = \frac{1}{L}\int_0^L f(\mathrm{r})\,dx_k\,, \quad \{f(\mathrm{r})\}_S = \frac{1}{S}\iint_{(S)} f(\mathrm{r})\,dx_i\,dx_j\,.$$

In this case, the following condition is satisfied

$$\left\{\{f(\mathrm{r})\}_S\right\}_L = \left\{\{f(\mathrm{r})\}_L\right\}_S = \langle f(\mathrm{r})\rangle\,.$$

The determination of the estimates of the upper and lower bounds of the thermoelastic properties by means of the method of integral cross-sections is based [8] on two techniques of arbitrary division of a representative volumetric element (RVE) V (Fig. 1a) in the chosen direction (for example, along the $0x_3$ axis) into differential volumetric elements (DVE): cylindrical DVE with base area dx_1dx_2 (Fig. 1b) and laminated DVE of thickness

Odessa State Polytechnic University, Odessa, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 70, No. 5, pp. 807-813, September-October, 1997. Original article submitted July 9, 1996.

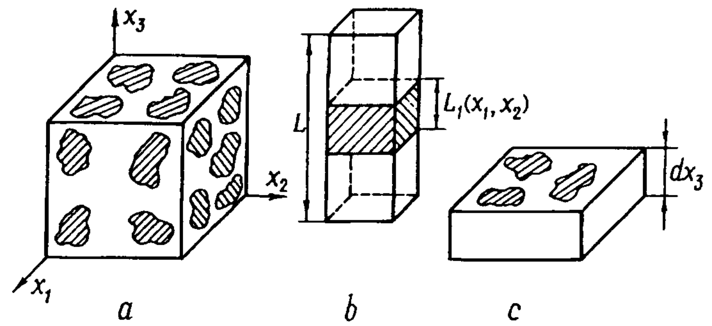

Fig. 1. Structure of microinhomogeneous material: a) representative volume; b) differential volumetric element (DVE) in the form of a prism; c) DVE in the form of a layer.

dx_3 (Fig. 1c). Moreover, each technique of division is associated with the selection of a pair of fictitious functions for stresses and deformations $\sigma'_{ij}(\mathrm{r})$ and $\varepsilon'_{ij}(\mathrm{r})$.

In the first technique of dividing the representative volumetric element into cylindrical differential volumetric elements it is assumed that the mean stress $\sigma'_{ij}(\mathrm{r})$ along the cross-section of the specimen $\{\sigma'_{ij}(\mathrm{r})\}_S$ perpendicular to the direction selected is equal to the volume-mean stress $\langle\sigma'_{ij}\rangle$, i.e., the following condition is satisfied

$$\left\{\sigma'_{ij}(\mathrm{r})\right\}_S = \langle\sigma'_{ij}\rangle, \quad \langle\varepsilon'_{ij}\rangle = \langle\varepsilon_{ij}\rangle. \tag{1}$$

In the second technique of dividing the representative volumetric element it is assumed that the mean deformation $\varepsilon'_{ij}(\mathrm{r})$ along length L in the selected direction $\{\varepsilon'_{ij}(\mathrm{r})\}_L$ is equal to the volume-mean deformation V of the representative volumetric element, i.e.,

$$\left\{\varepsilon'_{ij}(\mathrm{r})\right\}_L = \langle\varepsilon'_{ij}\rangle, \quad \langle\sigma'_{ij}\rangle = \langle\sigma_{ij}\rangle. \tag{2}$$

Let $H_{klij}(x_k)$ be the elasticity modulus tensor of a layer of thickness dx_k located perpendicularly to the $0x_3$ axis (Fig. 1):

$$H_{klij}(x_b) = C_{klmn} I_{ijmn} + \bar{S}_1(x_3)\,(C^{(1)}_{klmn} - C^{(2)}_{klmn})\,A^{(1)}_{mnij}(x_3), \tag{3}$$

where $\bar{S}_1(x_3)$ is the area of the intersection of the representative volumetric element by the plane $x_3 = \mathrm{const}$ occupied by the first component:

$$\bar{S}_i(x_3) = S_i(x_3)/S(x_3), \quad i = 1, 2; \quad S(x_3) = S_1(x_3) + S_2(x_3);$$

$$I_{ijmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}); \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The tensor A_{mnij} is defined from the equation

$$\left\{\varepsilon^{(1)}_{kl}(\mathrm{r})\right\}_{S_1} = A^{(1)}_{klmn}(x_3)\left\{\varepsilon_{mn}(\mathrm{r})\right\}_S.$$

Let $M_{klij}(x_1, x_2)$ be the compliance tensor of a prism of height L with base area dx_1dx_2:

$$M_{klij}(x_1, x_2) = S_{klmn} I_{mnij} + L_1(x_1, x_2)\,(S^{(1)}_{klmn} - S^{(2)}_{klmn})\,B^{(1)}_{mnij}(x_1, x_2), \tag{4}$$

where $L_1(x_1, x_2)$ is the length of the straight line that passes parallely to the $0x_3$ axis through the representative volumetric element over the first component:

$$\bar{L}_i(x_1, x_2) = L_i(x_1, x_2)/L(x_1, x_2), \quad i = 1, 2;$$

$$L(x_1, x_2) = L_1(x_1, x_2) + L_2(x_1, x_2).$$

The tensor $B_{mnij}(x_1, x_2)$ is defined from the equality

$$\left\{\sigma_{mn}^{(1)}(\mathbf{r})\right\}_{L_1} = B_{mnkl}^{(1)}(x_1, x_2)\left\{\sigma_{kl}(\mathbf{r})\right\}_L.$$

In order to determine the fork of the possible values of the effective elasticity modulus C_{ijkl}, we prove the following theorem.

Theorem. *If for two arbitrary pairs of functions* $(\varepsilon_{kl}, \sigma_{kl})$ *and* $(\varepsilon'_{kl}, \sigma'_{kl})$, *which characterize the state of an elastic body, the following inequalities are satisfied*

$$\sigma_{kl}\varepsilon_{kl} \le \sigma'_{kl}\varepsilon'_{kl} + 2\sigma_{kl}(\varepsilon_{kl} - \varepsilon'_{kl}), \tag{5}$$

$$\sigma_{kl}\varepsilon_{kl} \le \sigma'_{kl}\varepsilon'_{kl} + 2\varepsilon_{kl}(\sigma_{kl} - \sigma'_{kl}) \tag{6}$$

and for a pair of fictitious functions of the stresses $\sigma'_{ij}(\mathbf{r})$ *and deformations* $\varepsilon'_{ij}(\mathbf{r})$ *the following expressions are valid*

$$\langle\varepsilon'_{kl}\rangle = \left\{[M_{klij}(x_1, x_2)]^{-1}\right\}_S^{-1}\langle\sigma_{ij}\rangle \quad \textit{subject to condition} \quad (1),$$

$$\langle\sigma'_{kl}\rangle = \left\{[H_{klij}(x_3)]^{-1}\right\}_L^{-1}\langle\varepsilon_{ij}\rangle \quad \textit{subject to condition} \quad (2),$$

then there is the following inequality

$$\left\{[M_{klij}(x_1, x_2)]^{-1}\right\}_S \le C_{klij} \le \left\{[H_{klij}(x_3)]^{-1}\right\}_L^{-1}. \tag{7}$$

Proof. To determine the fork of the possible values of the effective elasticity modulus C_{ijkl}, we determine the connection of $\langle\sigma'_{kl}\rangle$ with $\langle\varepsilon_{kl}\rangle$ and of $\langle\varepsilon'_{kl}\rangle$ with $\langle\sigma_{kl}\rangle$. If we take into consideration that generally the equality

$$\left\{\sigma'_{kl}\right\}_S = \left\{C_{klij}(\mathbf{r})\cdot\varepsilon_{ij}(\mathbf{r})\right\}_S,$$

is satisfied, then on the basis of the linearity of the elasticity problem it is possible to write

$$\left\{\sigma'_{kl}\right\}_S = H_{klij}(x_3)\cdot\left\{\varepsilon_{ij}(\mathbf{r})\right\}_S, \tag{8}$$

where $H_{klij}(x_3)$ is defined in Eq. (3).

Multiplying Eq. (8) by the inverse tensor $[H_{klij}(x_3)]^{-1}$ on the left and then averaging the resulting expression over length L (variable x_3) with account for Eq. (1), we obtain

$$\langle\varepsilon_{ij}\rangle = \left\{[H_{klij}(x_3)]^{-1}\right\}_L\langle\sigma'_{kl}\rangle.$$

Thus,

$$\langle\sigma'_{kl}\rangle = \left\{[H_{klij}(x_3)]^{-1}\right\}_L^{-1}\langle\varepsilon_{ij}\rangle. \tag{9}$$

Now we will consider $\{\varepsilon'_{kl}(\mathrm{r})\}_L$. From the linearity of the elasticity problem we have

$$\left\{\varepsilon'_{kl}\right\}_L = M_{klij}(x_1, x_2)\left\{\sigma_{ij}(\mathrm{r})\right\}_L, \tag{10}$$

where $M_{klij}(x_1, x_2)$ is defined in Eq. (4).

Multiplying Eq. (10) by the inverse tensor $[M_{klij}(x_1, x_2)]^{-1}$ on the left and then averaging the resulting expression over the cross-section S perpendicular to the $0x_3$ axis and allowing for Eq. (2) we obtain

$$\langle\sigma_{ij}\rangle = \left\{[M_{klij}(x_1, x_2)]^{-1}\right\}_S \langle\varepsilon'_{kl}\rangle,$$

whence it follows that

$$\langle\varepsilon'_{kl}\rangle = \left\{[M_{klij}(x_1, x_2)]^{-1}\right\}_S^{-1} \langle\sigma_{ij}\rangle. \tag{11}$$

If instead of $(\sigma_{kl}, \varepsilon_{kl})$ and $(\sigma'_{kl}, \varepsilon'_{kl})$ we substitute the pairs $(\langle\sigma_{kl}\rangle, \langle\varepsilon_{kl}\rangle)$ and $(\langle\sigma'_{kl}\rangle, \langle\varepsilon_{kl}\rangle)$ into Eq. (5) and the pairs $(\langle\sigma_{kl}\rangle, \langle\varepsilon_{kl}\rangle)$ and $(\langle\sigma_{kl}\rangle, \langle\varepsilon'_{kl}\rangle)$ into Eq. (6), we obtain

$$\langle\sigma_{kl}\rangle\langle\varepsilon_{kl}\rangle \le \langle\sigma'_{kl}\rangle\langle\varepsilon_{kl}\rangle, \tag{12}$$

$$\langle\sigma_{kl}\rangle\langle\varepsilon_{kl}\rangle \le \langle\sigma_{kl}\rangle\langle\varepsilon'_{kl}\rangle. \tag{13}$$

Substituting (11) into (13) and (9) into (12) with account for the fact that the effective elasticity modulus C_{ijkl} is determined from the equation

$$\langle\sigma_{ij}\rangle = C_{ijkl}\langle\varepsilon_{kl}\rangle,$$

and rearranging, we obtain the expression

$$\left\{[M_{klij}(x_1, x_2)]^{-1}\right\}_S \le C_{klij} \le \left\{[H_{klij}(x_3)]^{-1}\right\}_L,$$

which was to be proved.

On the basis of this theorem, when the components of the microinhomogeneous materials are isotropic, we obtained the following formulas for the upper and lower bounds of Young's elasticity modulus E and Poisson's coefficient ν: the upper bound

$$E_{\mathrm{up}} = E_1\left[\left\{B_{11}\right\}_L - \nu_1\left\{B_{12}\right\}_L\right]^{-1}, \quad \left(\frac{\nu}{E}\right)_{\mathrm{up}} = \frac{1}{E_1}\left[\nu_1\left\{B_{11}\right\}_L - \left\{B_{12}\right\}_L\right], \tag{14}$$

where

$$B_{11} = \frac{1}{E_2}\frac{M - \nu_2 F}{M^2 - F^2}; \quad B_{12} = \frac{1}{E_2}\frac{F - \nu_2 M}{M^2 - F^2};$$

$$M = E_1^{-1}E_2^{-1}\left\{E\right\}_S; \quad F = \frac{\nu_1\nu_2}{E_1E_2}\left\{\frac{E}{\nu}\right\}_S;$$

the lower bound

$$E_{low} = \{\eta\}_S, \quad \left(\frac{\nu}{E}\right)_{low} = \frac{\nu_1}{E_1} - \frac{1-\nu_1}{E_1}\{B_{13}\}_S, \tag{15}$$

where

$$\eta = \left[\left\{\frac{1}{E}\right\}_L - 2\bar{L}_1(1-\bar{L}_1)\frac{\left(\frac{\nu_1}{E_1} - \frac{\nu_2}{E_2}\right)^2}{M-F}\right]^{-1}.$$

Here and after E_1 and E_2 are the Young moduli of component 1 (inclusion) and component 2 (matrix), respectively; ν_1 and ν_2 are the Poisson coefdficients of the first and second components, respectively.

Bending of an Inhomogeneous Plate. We consider an elastic problem concerning the bending of an inhomogeneous square plate ($0 \le x, y \le a$) of constant thickness h that consists of two different materials in two corresponding regions. For setting and solving the problem, we use a mathematical model of the bending of a Kirchhoff plate. The symmetry center of the plate is $x = a/2$, $y = a/2$.

The material that comprises the inner region of the plate G_1 (inclusion):

$$\frac{a}{2} - \varepsilon \le x, \quad y \le \frac{a}{2} + \varepsilon,$$

where $0 < \varepsilon < a/2$, has cylindrical stiffness D_1.

The material that comprises the outer region of the plate G_2 has cylindrical stiffness D_2.

For definiteness of the problem, we assume that the lateral edges of the plate are fixed in a rigid base. On the upper face of the plate there is an intensity load $g = g(x, y)$ normal to the plate. It is required to find the effective cylindrical stiffness D_{ef} of the given inhomogeneous plate for various geometric parameters of inclusion, as well as to determine the upper and lower bounds of the effective stiffness D_{ef} for the set of problems under consideration.

The boundary-value problem set is reduced to solution of the differential equation for a bent middle surface of the plate (to the C. Germen–Lagrange equation) [9]

$$Lw = \frac{\partial^4 w(x,y)}{\partial x^4} + 2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y)}{\partial y^4} = \begin{cases} \dfrac{g(x,y)}{D_1}, & \text{if } (x,y) \subset G_1 \\ \dfrac{g(x,y)}{D_2}, & \text{if } (x,y) \subset G_2 \end{cases} \tag{16}$$

or

$$\text{or} \quad \nabla^2\nabla^2 w = \frac{g(x,y)}{D_i},$$

where $D_i = E_i h^3 (12(1-\nu_i^2))^{-1}$, $i = 1, 2$ is the cylindrical stiffness of the plate; w is the displacement of the point along the z axis.

In our case, $g(x, y) = g = \text{const}$.

The continuity condition of displacements is fulfilled at the phase interface $B = G_1 \cap G_2$.

The boundary conditions for the given problem will be written as follows:

$$w|_{x=0,a} = \left.\frac{\partial w(x,y)}{\partial x}\right|_{x=0,a} = 0; \quad w|_{y=0,a} = \left.\frac{\partial w(x,y)}{\partial y}\right|_{y=0,a} = 0. \tag{17}$$

On the basis of Eqs. (14) and (15) it was proved that:

1) if the stiffness of the matrix is smaller than that of the inclusion ($D_1 < D_2$), then the lower and upper estimates of the effective stiffness in problem (16) and (17) will be determined in the following way: the upper bound

$$D_{up} = D_2 D' / (\bar{l}_2 D' + \bar{l}_1 D_2)\,, \tag{18}$$

where $D' = \bar{l}_1 D_1 + \bar{l}_2 D_2$; the lower bound

$$D_{low} = \left(\frac{\bar{l}_1}{D_1} + \frac{\bar{l}_2}{D_2} - f_2 \right)^{-1}, \tag{19}$$

where $f_2 = 2\bar{l}_1\bar{l}_2 \nu^2 (D_2 - D_1)^2 [D_1 D_2 (1 - \nu)(\bar{l}_1 D_2 + \bar{l}_2 D_1)]^{-1}$;

2) if the stiffness of the matrix is greater than that of the inclusion ($D_1 > D_2$), then the lower and upper estimates of the effective stiffness in problem (6)-(7) will be determined as follows: the upper bound

$$D_{up} = \bar{l}_1 D_1 + \bar{l}_2 D_2\,, \tag{20}$$

the lower bound

$$D_{low} = \bar{l}_2 D_2 + \bar{l}_1 D'' + f_2^0\,, \tag{21}$$

where

$$D'' = \left(\frac{\bar{l}_1}{D_1} + \frac{\bar{l}_2}{D_2} - f_2 \right)^{-1}; \quad f_2^0 = 2\,\frac{\bar{l}_1 \bar{l}_2 \nu^2 (D_2 - D'')^2}{D'' D_2 (1 - \nu)(\bar{l}_1 D_2 + \bar{l}_2 D'')}\,.$$

Here $\nu_1 = \nu_2 = \nu$; $\bar{l}_i = l_i / l$, l_i is the rib length of the i-th component, $i = 1, 2$, $l = l_1 + l_2$.

We will construct a finite-difference scheme for Eqs. (16)-(17) in the following manner. Using the straight lines $x = kc$, $y = kc$, $k = 1, 2, \ldots, n$, we construct a rectangular grid B_n that will divide the region $G = G_1 \cup G_2$ into $(n + 1)^\alpha$ cubes (α is the dimensionality of the region G).

There will be three types of cubes:

1) $V_{1k} \subset G_1$ (stiffness D_1);
2) $V_{2k} \subset G_2$ (stiffness D_2);
3) $V_{3k} \supset B$ (stiffness D_3).

Then, we replace each cube by a node (the node is located at the center of the cube) with connections that are determined by the stiffness properties in mutually perpendicular directions.

The stiffness of connections between adjacent nodes that belong to different types of cubes is determined as the harmonic mean of the stiffnesses of these cubes.

To determine stiffness D_3, we use Eq. (7). According to the method of integral cross-sections, stiffness D_3 will be determined as the arithmetic mean of the upper and lower bounds according to formulas (18)-(21).

As a result of such a construction we obtain a new grid B_n' whose nodes are joined by three types of connections. Taking into account that $D = D(x, y)$, the difference approximation of differential operator (16) can be represented in the following form: the like derivatives are

$$\left(\frac{\partial^4 (Dw)}{\partial x^4} \right)_{ij} = \frac{1}{c^4} (w_{i-2,j} D_{i-2,i-1;j} - w_{i-1,j} (D_{i-2,i-1;j} + 3D_{i-1,i;j}) +$$

$$+ w_{ij} (3D_{i-1,i;j} + 3D_{i,i+1;j}) - w_{i+1;j} (3D_{i,i+1;j} + D_{i+2,i+1;j}) + w_{i+2,j} D_{i+2,i+1;j})\,. \tag{22}$$

(the derivative with respect to y is determined in a similar fashion), the mixed derivatives are

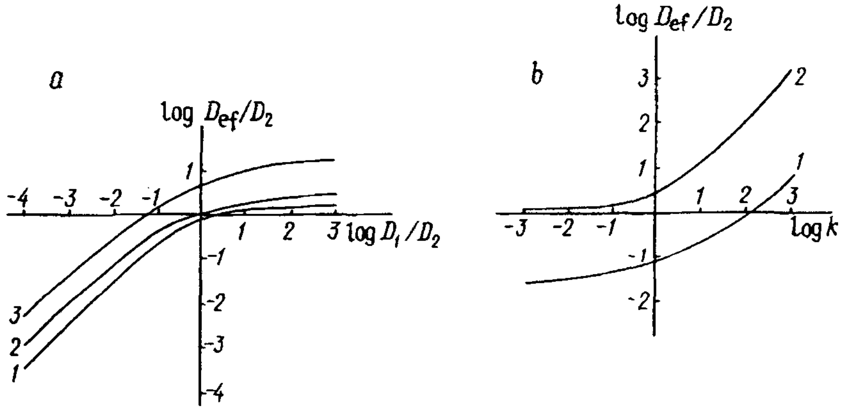

Fig. 2. Logarithmic dependence of D_{ef}/D_2 on the logarithm of D_1/D_2 (a) and the logarithm of k (b): a) 1, $k = 1/10$; 2) 10^0, 3) 10; b) 1, $D_2/D_1 = 100$; 2) 1/100.

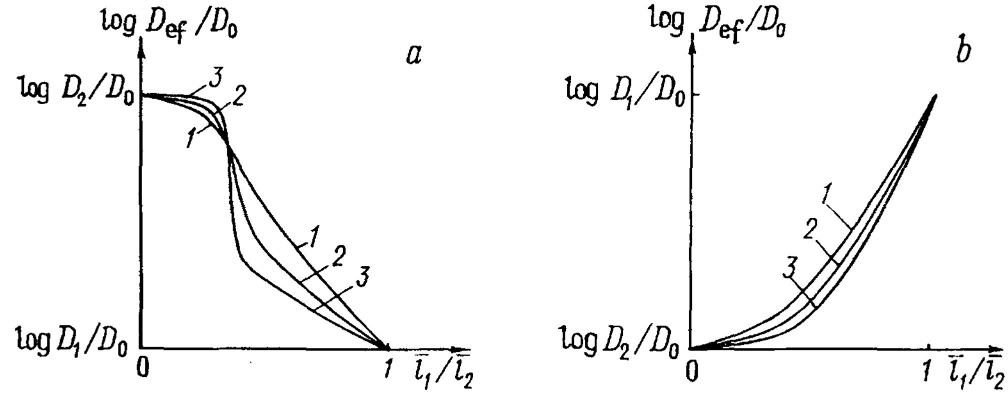

Fig. 3. Lograithmic dependence of D_{ef}/D_0 on l_1/l_2: a, 1) $D_1/D_2 = 1/10$, 2) 1/100, 3) 1/1000; b, 1) $D_1/D_2 = 10$, 2) 100, 3) 1000. $D_0 = 1$ n.m.

$$\left(\frac{\partial^4 (Dw)}{\partial x^2 \partial y^2}\right)_{ij} = \frac{1}{c^4}\Bigg[w_{ij}\left(D_{i-1,i;j} + D_{i+1,i;j} + D_{i;j+1,j} + D_{i;j-1,j}\right) -$$

$$- w_{i-1,j}\left(D_{i-1,i;j} + \frac{1}{2} D_{i-1;j,j-1} + \frac{1}{2} D_{i-1;j,j+1}\right) -$$

$$- w_{i+1,j}\left(D_{i+1,i;j} + \frac{1}{2} D_{i+1;j,j+1} + \frac{1}{2} D_{i+1;j,j-1}\right) -$$

$$- w_{i,j+1}\left(D_{i;j,j+1} + \frac{1}{2} D_{i-1,i;j+1} + \frac{1}{2} D_{i+1,i;j+1}\right) +$$

$$+ w_{i-1,j-1}\frac{1}{2}\left(D_{i-1;j-1,j} + D_{i-1,i;j-1}\right) + \frac{1}{2} w_{i+1,j+1}\left(D_{i+1;j+1,j} + D_{i+1,i;j+1}\right) +$$

$$+ \frac{1}{2} w_{i-1,j+1}\left(D_{i-1,i;j+1} + D_{i-1;j+1,j}\right) + \frac{1}{2} w_{i+1,j-1}\left(D_{i+1,i;j-1} + D_{i+1;j-1,j}\right)\Bigg]. \qquad (23)$$

Substituting Eqs. (22) and (23) into original differential equation (16) with allowance for boundary conditions (17), we come to a system of linear algebraic equations. Solving this system, we find the unknown deflections of the plate. Then, using the results obtained, we determine the effective cylindrical stiffness of the given inhomogeneous plate from the following formula:

TABLE 1. Results of Calculations of the Upper D_{up} and Lower D_{low} Bounds for Effective Stiffness D_{ef} with Different Relationship Between Side Lengths of Matrix l_m and Inclusion l_{inc}

D_1	D_2	$l_m:l_{inc}=9:2$			$l_m:l_{inc}=9:4$			$l_m:l_{inc}=9:5$		
		D_{ef}	D_{low}	D_{up}	D_{ef}	D_{low}	D_{up}	D_{ef}	D_{low}	D_{up}
1	1	1	1	1	1	1	1	1	1	1
1	5	4.7	2.6	4.8	3	1.8	4	2.3	1.6	3.5
1	10	9.2	3.3	9.5	4.2	2	7.2	3.1	1.7	6.4
1	100	91	4.3	94	7.9	2.2	74	5.2	1.8	60
1	1000	910	4.5	940	8.8	2.2	740	5.6	1.8	590
5	1	1.1	1	1.9	1.4	1.2	2.8	1.9	1.4	3.2
10	1	1.1	1.1	3	1.6	1.3	5	2.2	1.6	6
100	1	1.1	1.1	23	1.8	1.3	45	2.9	1.7	56
1000	1	1.1	1.1	220	1.9	1.4	440	3	1.7	560

$$D_{ef} = (n+1)^{-\alpha} \sum_{i=1}^{(n+1)^{\alpha}} L^{-1} w_{ij} g_i ,$$

where Lw is defined in Eq. (17).

The results of calculations are represented in Figs. 2 and 3. For calculations, we selected the following values: $h = 0.01$, $a = 1.8$, and $c = 0.3$.

As is seen from Fig. 2a, the quantitative analysis indicates that in the case of a "stiffer" matrix, compared to the inclusion, the interphase connection exerts a substantial influence on the effective stiffness of the inhomogeneous plate.

On the basis of the numerical results (Fig. 2b), we find that beginning from the value $k = 0.1$, its further increase considerably affects the growth of the effective stiffness of the inhomogeneous plate at any relationship between the stiffnesses of the matrix and inclusion.

Analysis of the numerical results given in Fig. 3 also allows the following conclusion: growth of the geometric parameters of the inclusion has a substantial effect on the effective stiffness of the inhomogeneous plate when the stiffnesses of the matrix and inclusion differ by more than an order of magnitude.

The lower D_{low} and upper D_{up} bounds of the effective stiffness of the plate D_{ef} were calculated by formulas (18)-(21) (see Table 1).

Conclusion. Using the method of integral cross-sections, an elastic problem concerning the bending of a Kirchhoff inhomogeneous square plate is solved. The estimates of the upper and lower bounds obtained for the effective elasticity modulus were applied for describing the phase interfaces in calculations of deflections of the inhomogeneous plate. It is shown that the conditions of interaction between the inclusion and matrix exert a substantial effect on the effective characteristics. The dependences of the effective stiffness of the inhomogeneous plate on the geometry and stiffness characteristics of the matrix and inclusion are obtained.

NOTATION

σ_{ij}, stress tensor; ε_{ij}, deformation tensor; k, cofficient of interaction between a matrix and inclusion; c, spacing in finite-difference scheme. Subscripts: low, lower; up, upper; ef, effective; m, matrix; inc, inclusion; A, body in state A; B, body in state B; S, cross-sectional area of representative volumetric element; L, rib length of representative volumetric element.

REFERENCES

1. V. L. Rvachev and A. P. Slesarenko, Logic Algebra and Integral Transformations in Boundary-Value Problem [in Russian], Kiev (1976).
2. N. S. Bakhvalov and G. P. Panasenko, Averaging of Processes in Periodic Media [in Russian], Moscow (1984).
3. O. Zenkevich and K. Morgan, Finite Elements and Approximations [in Russian], Moscow (1986).
4. G. N. Dul'nev and V. V. Novikov, Transfer Processes in Inhomogeneous Media [in Russian], Leningrad (1991).
5. V. V. Novikov and O. B. Papkovskaya, Inzh.-Fiz. Zh., 68, No. 2, 322-329 (1995).
6. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Moscow (1978).
7. J. Webman, Z. Jorther, and M. N. Cohen, Phys. Rev., 13, No. 2, 213-224 (1976).
8. V. A. Malyshev and P. V. Malyshev, Dokl. Akad. Nauk USSR, Ser. A, No. 12, 48-53 (1987).
9. A. N. Podgornyi, G. A. Marchenko, and V. I. Pustynnikov, Foundations and Methods of Applied Elasticity Theory [in Russian], Kiev (1981).